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Conservation Form of the Navier-Stokes Equations in General Nonsteady Coordinates

Z.U.A. Warsi*
Mississippi State University, Mississippi State, Miss.

Introduction

RECENT interest in the generation of general body-oriented curvilinear coordinate systems,[1-3] for the purpose of solving the complete Navier-Stokes system of equations for subsonic and supersonic[4] flows, has given rise to many forms of presentations of the equations both in conservative and nonconservative formulations. Based on the available solutions of the gasdynamic equations (e.g., Ref. 5) the conservation-law form of the equations seems definitely preferable, particularly when shocks are present. Although the above statement cannot be repeated in a definitive sense for the case of viscous subsonic and supersonic flow prediction through the Navier-Stokes equations, nevertheless, it is expected that the conservation-law form may eventually be more acceptable for numerical purposes.

The purpose of this paper is to derive the conservation-law form of the Navier-Stokes equations in general nonsteady coordinate systems in a simple and direct fashion. Previous work on this subject has been done by McVittie,[6] Viviand,[7] and Vinokur.[8] It will be shown in this Note that the equations in the conservation-law form can be obtained simply by a little manipulation of some standard vector and tensor formulas.

Analysis

The general conservation law for classical fields in integral form is stated as[9,10]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V A\,\mathrm{d}\nu+\int_S \boldsymbol{B}\cdot\boldsymbol{n}\,\mathrm{d}s=\int_V C\,\mathrm{d}\nu \tag{1}$$

where $\boldsymbol{n}$ is the unit outward normal to the surface S which encloses the material volume $V(t)$. The integral law [Eq. (1)] states that for a given material volume $V(t)$ bounded by the material surface $S(t)$, the rate of what is contained in V at time t plus the rate of what flows out of S is equal to what is furnished to V. The quantities A, B, and C are tensor quantities such that A and C have the same tensorial order, while if $B\neq 0$ then it is a one-order higher tensor than A. Using Reynold's transport theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V A\,\mathrm{d}\nu=\int_V \frac{\partial A}{\partial t}\,\mathrm{d}\nu+\int_S A\,(\boldsymbol{v}\cdot\boldsymbol{n})\,\mathrm{d}s$$

where $\boldsymbol{v}$ is the continuum velocity vector, the conservation law takes the form

$$\int_V \frac{\partial A}{\partial t}\,\mathrm{d}\nu+\int_S \boldsymbol{f}\cdot\boldsymbol{n}\,\mathrm{d}s=\int_V C\,\mathrm{d}\nu \tag{2}$$

or

$$\int_V\left[\frac{\partial A}{\partial t}+\operatorname{div}\boldsymbol{f}-C\right]\mathrm{d}\nu=0 \tag{3}$$

where

$$\boldsymbol{f}=A\boldsymbol{v}+\boldsymbol{B}$$

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*Professor, Dept. of Aerospace Engineering.

Equation (3) in differential form is

$$\frac{\partial A}{\partial t}+\operatorname{div}\boldsymbol{f}=C \tag{4}$$

where C is the source term.

From a computational point of view we state that: for smooth solutions, if the difference approximations of Eqs. (2) and (4) are exactly the same expressions, then the conservation law is preserved. Moreover, since jump relations are implicit within the conservation form of the equations, the above statement remains valid for weak solutions, provided that Eq. (4) has been written in the strong conservation-law form for curvilinear coordinates—which is the subject matter of this paper.

As is well known, when Eq. (4) is written in the rectangular Cartesian coordinates, the conservation law is exactly satisfied. If, on the other hand, the coordinate transformation is curvilinear and $\boldsymbol{f}$ is a second- or higher order tensor, then some undifferentiated terms appear. Thus the main effort to preserve conservation in general coordinates is to write Eq. (4) in a form so that undifferentiated terms do not appear.

Equations in Steady and Nonsteady Coordinates

In fluid flow problems the Cartesian coordinates $(x_i)=\boldsymbol{r}$ represent a fixed point of the physical space. Thus no matter how the flow takes place, the time derivative of $\boldsymbol{r}$ in the sense of Eulerian variables is zero. That is

$$\left(\frac{\partial \boldsymbol{r}}{\partial t}\right)_{(x_i)}=0 \tag{5}$$

Let F be a scalar or vector flow quantity. A time-independent transformation

$$\xi^j=\xi^j(x_i) \tag{6}$$

yields the time derivative

$$\left(\frac{\partial F}{\partial t}\right)_{(x_i)}=\left(\frac{\partial F}{\partial t}\right)_{(\xi^j)} \tag{7}$$

provided the inverse of the transformation [Eq. (6)] exists. On the other hand, if the coordinate transformation is time dependent, viz.,

$$\xi^j=\xi^j(x_i,t) \tag{8}$$

then

$$\left(\frac{\partial F}{\partial t}\right)_{(x_i)}=\left(\frac{\partial F}{\partial t}\right)_{(\xi^j)}+(\operatorname{grad}F)\cdot\boldsymbol{w} \tag{9}$$

where $\boldsymbol{w}$ is a vector whose contravariant components are $\partial\xi^i/\partial t$. Thus Eq. (5) takes the form

$$\left(\frac{\partial \boldsymbol{r}}{\partial t}\right)_{(\xi^i)}+\boldsymbol{w}=0 \tag{10a}$$

where grad $\boldsymbol{r}=\tilde{I}$ is the idem tensor.

Before we proceed further it is important to list some important formulas from tensor theory.[11] The covariant base vectors are denoted by $\boldsymbol{a}_i$. Using the basis $\boldsymbol{a}_i$, and denoting the Christoffel symbols of the second kind by Γ^i_{jk}, we have†

$$\boldsymbol{a}_i\cdot\boldsymbol{a}_j=g_{ij} \tag{10b}$$

†Except in x_i and ξ^j the subscripts denote covariants while the superscripts denote contravariant components. Repeated indices always imply summation.

$$\sqrt{g}=a_i\cdot(a_j\times a_k)>0,\ i,j,k \text{ cyclic} \tag{10c}$$

$$a^i=\frac{1}{2\sqrt{g}}e^{ijk}(a_j\times a_k) \tag{10d}$$

$$\Gamma^i_{ir}=\frac{1}{2g}\frac{\partial g}{\partial x^r} \tag{10e}$$

$$\frac{\partial a_i}{\partial \xi^j}=\frac{\partial a_j}{\partial \xi^i}=\Gamma^\ell_{ij}a_\ell \tag{10f}$$

$$\frac{\partial g^{ij}}{\partial \xi^j}=-\Gamma^i_{\ell j}g^{\ell j}-\Gamma^j_{j\ell}g^{i\ell} \tag{10g}$$

$$\frac{\partial}{\partial \xi^i}(\sqrt{g}g^{ij})=-\sqrt{g}\Gamma^j_{\ell m}g^{\ell m} \tag{10h}$$

$$\frac{\partial a_2}{\partial t}\cdot(a_3\times a_1)=a_1\cdot\left(\frac{\partial a_2}{\partial t}\times a_3\right) \tag{11a}$$

$$\frac{\partial a_3}{\partial t}\cdot(a_1\times a_2)=a_1\cdot\left(a_2\times\frac{\partial a_3}{\partial t}\right) \tag{11b}$$

$$\text{div } A=\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^i}(\sqrt{g}A^i)\ \text{(divergence of a vector)} \tag{11c}$$

$$\text{div } \tilde{\tau}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^k}(\sqrt{g}\tau^{ik}a_i)\ \text{(divergence of a tensor)} \tag{11d}$$

[Note that the divergence of a tensor always has some undifferentiated terms. However, the introduction of base vectors a_i, as in Eq. (11d), alleviates this problem.]

Taking the divergence of w and using Eq. (11c), we have

$$\text{div } w=\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^i}\left(\sqrt{g}\frac{\partial \xi^i}{\partial t}\right) \tag{12a}$$

Now from Eq. (10a)

$$\frac{\partial r}{\partial t}=-a_j\frac{\partial \xi^j}{\partial t}$$

so that

$$\frac{\partial \xi^i}{\partial t}=-g^{ij}\frac{\partial r}{\partial t}\cdot a_j \tag{12b}$$

Using Eq. (12b) in Eq. (12a), we obtain

$$\text{div } w=-\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^i}\left[(\sqrt{g}g^{ij})\left(\frac{\partial r}{\partial t}\cdot a_j\right)\right] \tag{13}$$

Performing the differentiation on groups of terms shown in parentheses in Eq. (13) and using Eqs. (10d), (10f), (10h), (11a), and (11b), we get

$$\text{div } w=-\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}(\sqrt{g}) \tag{14}$$

Equation (14) is the fundamental equation for nonsteady coordinates.[12]

We shall now treat the three conservation laws of fluid dynamics separately.

1) Mass conservation:

Writing $A=\rho$ (density), $B=C=0$ and replacing the time derivative $(\partial\rho/\partial t)_{(x_i)}$ by the right-hand side of Eq. (9), we get on using Eq. (11c)

$$\frac{\partial}{\partial t}(\rho\sqrt{g})+\frac{\partial}{\partial \xi^i}(\rho\sqrt{g}u^i)=0 \tag{15}$$

where u^i are the contravariant components of a modified velocity vector defined by

$$u=v+w \tag{16}$$

2) Energy conservation:

Writing $A=\psi=\rho e+\frac{1}{2}\rho|v|^2$ (total enthalpy per unit volume)

$$B=\psi v+\tilde{\sigma}\cdot v-S_0\mu \text{ grad } T, C=0$$

and following the same approach as in obtaining Eq. (15), we get

$$\frac{\partial}{\partial t}(\psi\sqrt{g})+\frac{\partial}{\partial \xi^i}(\sqrt{g}b^i)=0 \tag{17}$$

where b^i are the contravariant components of the vector b defined as

$$b=\psi u+\tilde{\sigma}\cdot v-S_0\mu \text{ grad } T$$

3) Momentum conservation:

Writing

$$A=\rho v$$

$$B=\tilde{\sigma}$$

$$=p\tilde{I}-\frac{1}{Re}[(\lambda\text{div } v)\tilde{I}+\mu \text{ def } v]$$

$$C=0$$

$$\text{def } v=\text{grad } v+(\text{grad } v)^T$$

and replacing $(\partial/\partial t)(\rho v)$ by the right-hand side of Eq. (9) with $F=\rho v$ and using Eq. (16), we get an equation involving the divergences of ρuu, ρww, ρuw, and ρwu. Using the identity

$$\text{div}(AB)=(\text{grad } A)\cdot B+(\text{div } B)A$$

for div(ρww) and div(ρuw) and using Eq. (14), we get

$$\frac{\partial}{\partial t}(\rho\sqrt{g}v)+\sqrt{g}\text{div}[\rho vu+\tilde{\sigma}]=0 \tag{18}$$

On using Eq. (11d) in Eq. (18), we obtain

$$\frac{\partial}{\partial t}(\rho\sqrt{g}v^k a_k)+\frac{\partial}{\partial \xi^j}[(\rho v^i u^j+\sigma^{ij})\sqrt{g}a_i]=0 \tag{19}$$

Equations (15), (17), and (19) are the equations of fluid dynamics in the strong conservation law form for any nonsteady and general coordinate system ξ^i. For steady and general coordinates the equations retain the same form but $w=0$, viz., $u=v$.

For numerical computation the scalar form of the conservation equations is needed which can be easily obtained. For example, in a general curvilinear coordinate system (ξ^1,ξ^2,ξ^3), embedded in the rectangular Cartesian system (x_1,x_2,x_3), the base vectors a_k are given by

$$a_k=i_n\frac{\partial x_n}{\partial \xi^k}$$

where i_n are the constant unit vectors. Equating the coefficients of i_n $(n=1,2,3)$ in Eq. (19), we obtain the three momentum equations in scalar form.

Further analysis shows that in the case of axisymmetric flow, the scalar momentum equations (obtained from Eq. (19) with the corresponding expression of σ^{ij} substituted therein) cannot be stated in the strong conservation-law forms. In this form some undifferentiated terms appear which act as "sources" of momentum. Vinokur[8] has shown that the same terms appear when the conservation law [Eq. (3)] is applied to a finite-volume element for axisymmetric flow.

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Supersonic Flow Past Conical Bodies with Nearly Circular Cross Sections

Martin C. Jischke*
University of Oklahoma, Norman, Okla.

Introduction

DOTY and Rasmussen,[1] using a constant-density approximation, have presented accurate, analytical results for hypersonic flow past an inclined cone. Lee and Rasmussen,[2] employing similar approximations that linearize the governing equations, have analyzed hypersonic flow past elliptical cones. It is the purpose of this paper to extend this approach to supersonic flow past cones of more arbitrary cross section.

The analysis uses a straightforward perturbation of supersonic flow past an unyawed cone. As first noted by Ferri,[3] such a procedure fails for cones at yaw or with noncircular cross sections in a thin layer of intense vorticity near the cone called the vortical layer. The solutions obtained herein must, thus, be viewed as the first-order outer expansion in a matched asymptotic expansion scheme and must, in principle, be asymptotically matched to solutions valid in the vortical layer. Investigations by Munson[4] and others, however, have shown that the results obtained for the circumferential velocity and pressure by means of a regular perturbation scheme are uniformly valid throughout the shock layer, a conclusion that extends the use of the present paper.

Analysis

We employ a spherical polar coordinate system (r,θ,ϕ) with origin at the vertex of the cone which is assumed to be embedded in uniform supersonic stream along the cone axis. The cone is assumed to be sufficiently slender that the shock is attached to its pointed nose. The cone cross section is described by means of a Fourier series for the polar angle of the body,

$$\theta_c = \delta + \sum_{n=1} \epsilon_n \cos[n(\phi-\phi_n)]$$

Here δ is the half-angle of the basic circular cone and the ϵ_n parameters that describe the deviation of the cross section from a circle. For example, ϵ_2 measures elliptical eccentricity. The ϕ_n determine the relative phase of the various Fourier components. Our perturbation scheme assumes the ϵ_n are small compared to δ. This representation of the body shape suggests the following form for the shock angle θ_s, the radial velocity u, and the azimuthal velocity w,

$$\theta_s = \beta + \alpha g_0 \cos(\phi-\phi_0) + \sum_{n=1}^{\infty} \epsilon_n g_n \cos[n(\phi-\phi_n)] \tag{1}$$

$$u = u_0(\theta) + \alpha U_0(\theta)\cos(\phi-\phi_0) + \sum_{n=1}^{\infty} \epsilon_n U_n(\theta)\cos[n(\phi-\phi_n)] \tag{2}$$

$$w = \alpha W_0(\theta)\sin(\phi-\phi_0) + \sum_{n=1}^{\infty} \epsilon_n W_n(\theta) n \sin[n(\phi-\phi_n)] \tag{3}$$

The expansions for the polar velocity v, entropy s, sound speed a, and pressure p are identical in form to that for the radial velocity. Lower case quantities refer to the unyawed circular cone flow, a result that is presumed to be known. The angle of attack α is also assumed to be small compared to δ.

Substituting these expansions into the inviscid, adiabatic, compressible flow equations as well as the shock jump relations and tangency condition at the body and then equating powers of α and the ϵ_n to zero gives a hierarchy of problems, the first being that past an unyawed circular cone. The next problem corresponds to the linear effects of angle of attack α and noncircular cross sections ϵ_n and can, after some manipulation, be reduced to the following second-order ordinary differential equation for the radial velocity perturbation $U_n(\theta)$,

$$U_n'' + \cot\theta U_n' + (2 - n^2\csc^2\theta) U_n = n^2\csc^2\theta F_n(\theta) + G_n \tag{4}$$

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*Professor and Director, School of Aerospace, Mechanical, and Nuclear Engineering. Associate Fellow AIAA.